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Critical Phenomena and Finite-size Scaling in Communication Networks★

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***Abstract*— This paper presents a statistical mechanics-based approach to investigate critical phenomena and size scaling in communication networks. The qualitative nature of phase transitions in the underlying network systems is characterized; and its static and dynamic critical behaviors are identified. Effects of network size, different routing strategies have been analyzed. In all these analyses, phase transition is considered using a single intensive parameter of the communication network system, namely the external packet load. These problems have been investigated by extensive simulation on the model of a wired communication network.**

1. Introduction

A complex network is broadly defined as a collection of interconnected and interacting systems [1], [2] where the individual subsystems (or participating agents) themselves could be complex dynamical systems. Complex networks have been shown to characterize the behavior and topological organization of many natural and engineered systems, such as those found in the disciplines of sociology [3], biology [4], finance [5], communication networks [6], sensor networks etc. A common feature across all these multi-agent systems is that their global behavior emerges from local dynamics of the participating agents. Thus, dynamics of complex networks can be characterized by the relation between micro-motion (i.e., local dynamics) and macro-motion (i.e., global dynamics), which could be characterized by application of the principles of Statistical Mechanics [1], [7], [8].

A characteristic phenomenon of complex systems, consisting of interacting and interdependent dynamics, is *phase transition*, where an abrupt non-smooth change in the operating characteristics may take place with a relatively small variation of the system parameter(s). One of the prime uses of statistical mechanics in the field of networks is to characterize this critical phenomenon. For network communication, this phenomenon would correspond to dependence of the network's global characteristics (e.g., connectivity, average rate of change of queue length, average packet drop rate etc.) on local parameters (e.g., communication radius, packet load, transmission probability etc.) [9], [10], [11]. A key task in the analysis of phase transitions is to characterize the system behavior in the vicinity of a critical point. This paper focuses on analysis of congestion problems in communication networks as phase transitions or critical phenomena in the sense of statistical Mechanics. Typically, for communication networks, concept of bottleneck buffers (routers) is used to detect and control (i.e., mitigate) network congestion. Due to the use of bottleneck buffers, the network structure essentially reduces to a multi-source and single destination (or vice-versa) one, that helps developing average global analytical models (mean-field models) as shown in [12]. Due to same reason, mean field models of 1-D network or Cayley-tree network are also tractable that can be found in [11]. However, in general, multi-source and multi-destination communication problems need to be analyzed, which in turn makes the tasks of modeling and optimal routing much harder and often analytically intractable. From this perspective, tools of statistical mechanics prove to be very useful. It is also necessary to understand finite-size scaling laws to analyze the critical behavior of necessarily finite-size systems (e.g., communication networks) that are usually very small in size compared to typical thermodynamic systems. This paper presents a statistical mechanics-based approach to investigate critical phenomena and size scaling by extensive simulation on the model of a wired communication network. The network modeling and the choice of order parameter are based on [10] and [11], respectively. The major contributions of this article beyond the authors' recent work [9] in this direction are i) estimation of relaxation time to detect

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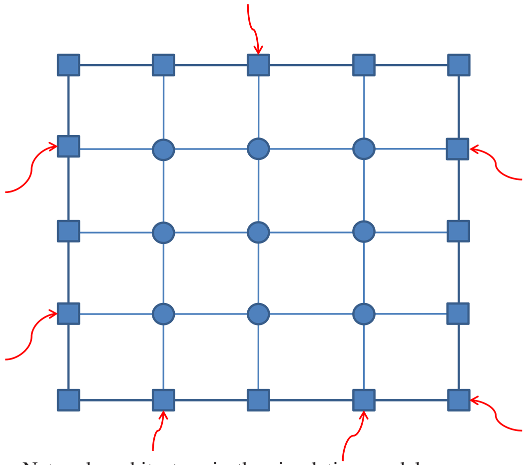

Fig. 1. Network architecture in the simulation model

phase transition in communication networks and ii) analysis of the effects of finite size and routing strategies on the phenomenon.

2. Phase Transition in Square-grid wired Network Communication

The network under consideration is a two-dimensional square grid as shown in Fig. 1, where the nodes (routers) are placed at the grid points. For a square grid network with $N \times N$ nodes there are $4 \times (N-1)$ boundary nodes (shown as squares in the Fig. 1) and $N^2 - 4 \times (N-1)$ internal nodes (shown as circles in the Fig. 1). Only boundary nodes are assumed to be the sources and/or the sinks for packet generation and termination; internal nodes can only transmit the packets. Each node receives packets in an infinite queue from its neighboring nodes and packets are terminated after reaching their destinations. In each time unit, packets are created in the boundary nodes with a Poisson arrival rate λ. Destination nodes are chosen randomly from the boundary nodes, including their source nodes. Each node transmits one packet from the head of its queue to a deterministically chosen neighboring node at each time unit. The node chosen to forward a data packet is selected so that the packet travels via the shortest path to its destination. When there are more than one candidate nodes for the shortest path, the node with a smaller queue length is chosen to prevent early congestion of the network.

In this context, the congestion phenomenon can be viewed as a continuous phase transition as discussed in [9], where the global intensive parameter that triggers the network phase transition is the packet arrival rate λ and the global order parameter is the rate of change of average queue length per node. However, as only $4 \times (N-1)$ boundary nodes generate packets and all N^2 nodes share these packets, a *surface correction* is needed to define the proper intensive parameter. To avoid the surface effect, an effective load per node, λ_{eff} is defined as follows:

$$\lambda_{eff} = \lambda \cdot \frac{4(N-1)}{N^2} \tag{1}$$

Similarly, to define the order parameter, scaling adjustments are made as discussed in the sequel. Let $Q(t)$ is the total queue length (total number of packets) in the network at time t. Thus, average queue length per node at time t is $q(t) = Q(t)/N^2$ and the rate of change of average queue length per node is modeled as:

$$\dot{q}(t) = \lim_{\Delta t \to 0} \frac{q(t) - q(t-\Delta t)}{\Delta t} \tag{2}$$

Then, the expected value of the time rate of change of average queue length, $\langle \dot{q}(t) \rangle$ is normalized to define the order parameter, M.

$$M = \frac{\langle \dot{q}(t) \rangle}{\lambda_{eff}} \tag{3}$$

In the above equation, M is the fraction of incoming packets that accumulate inside the network per unit time. Therefore, if there is no congestion, M becomes negligible, i.e. there is no packet accumulation. In the event of network congestion, the worst scenario could be zero packet departure, i.e., all packets that enter the network accumulate inside. In that case, M is approximately equal to 1. Following this procedure, M has been computed for given values of λ_{eff}. To eliminate the effect of transients, the expected value of $\dot{q}(t)$ is calculated from only steady-state time series data. The plot of M vs. λ_{eff} from a Monte Carlo simulation for a 10×10 network is depicted in Fig. 2; there is a critical value $\lambda^c_{eff} \approx 0.11$ of effective load per node such that, for $\lambda_{eff} < \lambda^c_{eff}$, the value of order parameter M is almost negligible. In contrast, for $\lambda_{eff} > \lambda^c_{eff}$, M takes on non-zero values. This abrupt change of system behavior across the critical value λ^c_{eff} is identified as a continuous *phase transition*, where the communication network moves from a *steady phase* of negligible M to an *unsteady phase* of finite positive M.

A phase transition is marked by the presence of analytical singularities or discontinuities in the functions describing macroscopic physical parameters of the system. In the vicinity of the critical point marking the phase transition, the functional form of the order parameter is often modeled by the *power law*, $m \sim |T - T_c|^\beta$ with *critical exponent* β, where m is the order parameter, T is the intensive variable (e.g., temperature) and T_c is the critical value of T corresponding to the phase transition. The log-scale plot in Fig. 3 shows that the critical exponent β is approximately 0.3. However, in the sense of statistical mechanics, β is defined only in the close vicinity of the critical point. Since it is very difficult to simulate a large system close to the critical point, there is apparently no reliable information on M in the vicinity of λ^c_{eff}. Thus, the plot given here makes use of values of M up to a certain proximity of λ^c_{eff}.

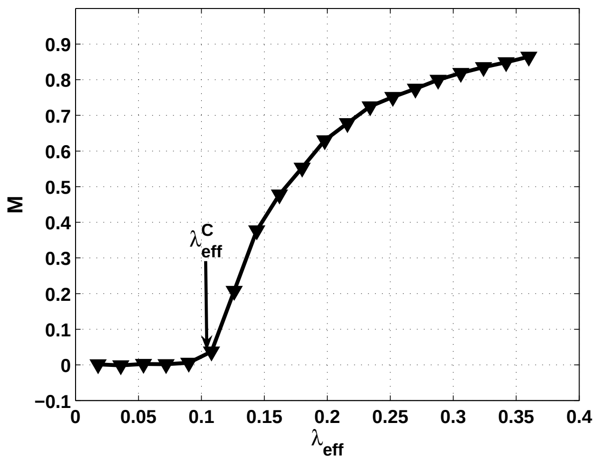

Fig. 2. Continuous phase transition in network communication

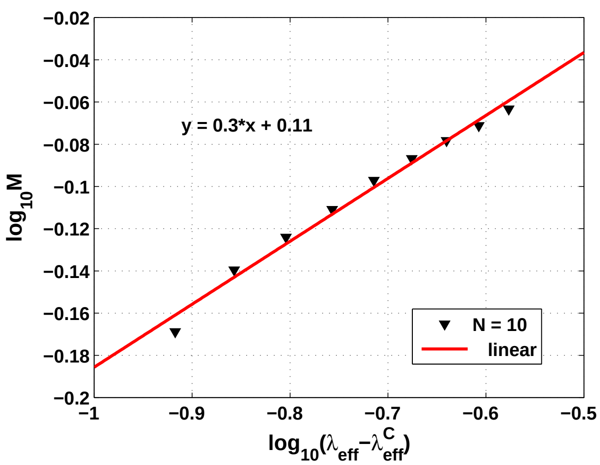

Fig. 3. Identification of critical exponent β

A. *Critical Slowing Down*

In statistical mechanics, it is known that the spatial correlation length ξ increases as the system approaches the critical point and, at the critical point, spatial correlation pervades the whole system. The process is modeled by the power law, $\xi(t) \sim |t|^{\nu}$ near the critical point, where $t = T - T_c$, and ν is a critical exponent that is $1/2$ according to Landau-Ginzburg theory. More accurate values that match closely to the experimental data, are obtained by other theories [13]. Thus, ν is one of the important static exponents. Similar to the spatial correlation properties, certain temporal correlation laws are also present for critical phenomena in large systems, which lead to the definition of *Dynamic Critical Exponent* [14]. Let Z be the dynamic critical exponent and τ be the relaxation time or the temporal correlation length of the slowest mode. Near a phase transition point, the relaxation time of the slowest mode of a system diverges as $\tau \sim \xi^{Z}$, Formally, the normalized time-correlation function $\phi_x(t_1 - t_2)$ of an observable x(t) is defined as [15]:

$$\phi_x(t_1 - t_2) = \frac{[\langle x(t_1)x(t_2)\rangle - \langle x\rangle^2]}{[\langle x^2\rangle - \langle x\rangle^2]} \tag{4}$$

with the boundary conditions, $\phi_x(0) = 1$ and $\phi_x(\infty) = 0$, and the function $\phi_x(t)$ decays monotonically with t. If ϕ_x is time integrable, then its integral, $\tau_x \triangleq \int_0^{\infty} \phi_x(t)dt$, is called the relaxation time or temporal correlation length of the observable x(t); in the sequel, τ_x is abbreviated as τ. Although more than one relaxation time may exist, the slowest mode is considered as:

$$\frac{[\langle x(t_1)x(t_2)\rangle - \langle x\rangle^2]}{[\langle x^2\rangle - \langle x\rangle^2]} \approx exp\left(-\frac{|t_1 - t_2|}{\tau}\right) \tag{5}$$

The increase of τ near the critical point is known as the *Critical slowing down* [13]. That is, near the phase transition point, the time interval between two consecutive independent observations become longer, which makes it difficult to simulate a large system at the phase transition point. On the other hand, if the relaxation time can be estimated, then the phase transition point can be anticipated. The concept of *Statistical Dependence time*, will be considered here to estimate the relaxation time. The procedure for estimation of *Statistical Dependence time* in a continuous-time form is briefly derived below. Let the observable $x(t, \zeta)$ be a stochastic process, where ζ denotes a random sample. To estimate the relaxation time τ of $x(t, \zeta)$, let the average process $s(\zeta)$ be defined as:

$$s(\zeta) = \frac{1}{T}\int_0^T x(t, \zeta)dt \tag{6}$$

where T is the length of the time window per sample. The variance of $s(\zeta)$ across the samples is obtained as:

$$\begin{aligned}
\sigma_s^2 &= \frac{1}{T^2}\int_0^T\int_0^T \left(E[x(t_1)x(t_2)] - \langle x\rangle^2\right) dt_1 dt_2 \\
&\approx \frac{\sigma_x^2}{T^2}\int_0^T\int_0^T e^{-\frac{|t_1-t_2|}{\tau}} dt_1 dt_2 \quad (using\ Eqn.\ 5) \\
&= \frac{\sigma_x^2}{T^2}\int_0^T\int_{-b}^{T-b} e^{-\frac{|a|}{\tau}} da db = \frac{\sigma_x^2}{\left(\frac{T}{2\tau_{dep}}\right)}
\end{aligned} \tag{7}$$

where $\sigma_x^2 \triangleq \langle x^2\rangle - \langle x\rangle^2$ and $\tau_{dep} \triangleq \tau[1 - \frac{\tau}{T} + \frac{\tau}{T}e^{-\frac{T}{\tau}}]$. If $x(t, \zeta)$ is an uncorrelated process (e.g., a white noise), then the variance of the average process decays as fast as the length T of the time window. However, if the process is positively correlated, then the decay rate of the variance changes to $\frac{T}{2\tau_{dep}}$. That is, a positive correlation decreases the rate of decay of the variance of the average process. For, $T \gg \tau$, it can easily be seen that $\tau \sim \tau_{dep}$ and

$$\tau \approx \frac{\sigma_s^2 T}{2\sigma_x^2} \tag{8}$$

In the present case, a very long time series of $\dot{q}(t)/\lambda_{eff}$ is collected. The time series is divided into several windows of length T. Assuming ergodicity, different windows can be

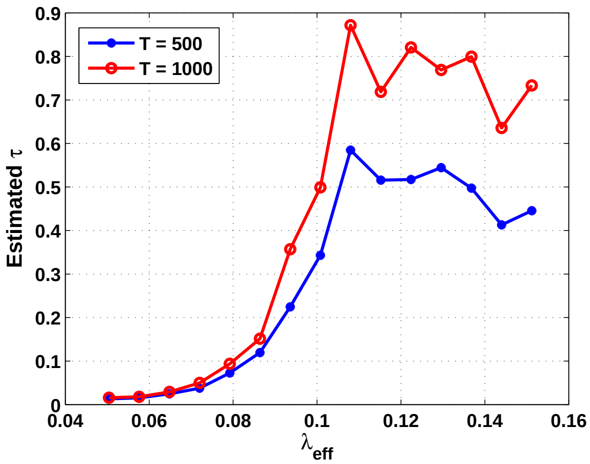

Fig. 4. Estimation of relaxation time τ

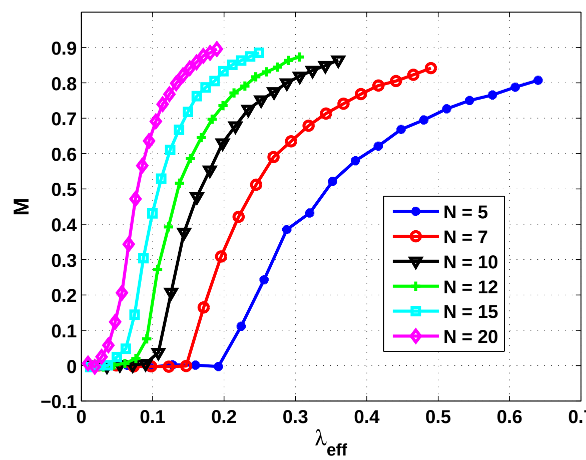

Fig. 5. Continuous phase transition plots for networks of different sizes

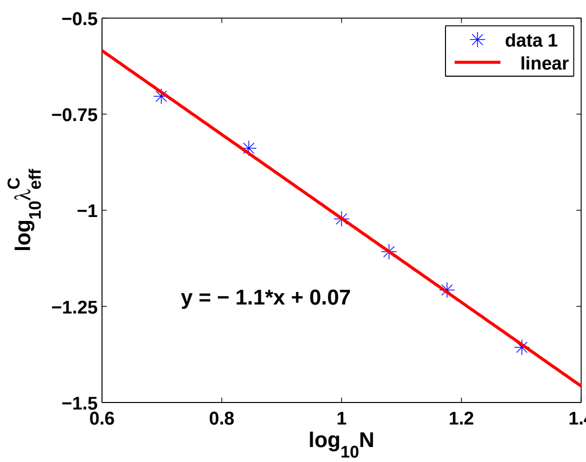

Fig. 6. Size dependency of critical network load

thought of as different samples. The mean is calculated for each sample and variance of these mean values would be σ_s^2 corresponding to the observable $\dot{q}(t)/\lambda_{eff}$. Estimated relaxation time τ is calculated for a 10×10 network in this way and the plot of τ vs. λ_{eff} is shown in Fig. 4 using $T = 500$ and $T = 1000$. As the system approaches the critical point, τ increases and a larger window length is required to achieve a smaller estimation error. This is seen in the plot of Fig. 4 where the difference in estimated values of τ with different window lengths becomes higher as the system approaches the critical point. This estimation procedure of τ is useful for dynamically detecting a critical phenomena in large complex systems; it simply needs to fix a threshold for the relaxation time.

3. Effect of Network Size

Human-engineered multi-agent systems are of significantly smaller size compared to natural thermodynamic systems that consist of very large numbers of particles. Therefore, size-scaling laws need to be carefully formulated for such human-engineered systems. In general, for finite-sized systems, the critical value of the intensive parameter, denoted as $T_c(N)$, is a function of the size N, and as N goes to infinity, $T_c(N)$ converges to $T_c(\infty)$ of the corresponding infinite-sized system. Figure 5 shows continuous-phase-transition plots for networks of different sizes - 5×5, 7×7, 10×10, 12×12, 15×15, and 20×20. The plot in Fig. 6 shows that indeed $\lambda^c_{eff}(N)$ decreases with increase in the network size N. It is already stated that the space correlation length $\xi(t)$ behaves as a function of $|t|$ in a power law. In the statistical mechanics literature, the following size dependency form of $\xi(t)$ is assumed [13], which conforms to the nature of $\xi(t)$ function.

$$\xi(T_c(N) - T_c(\infty)) = aN \tag{9}$$

where a is a constant parameter. The following law is now derived.

$$T_c(N) = T_c(\infty) + bN^{-\frac{1}{\nu}} \tag{10}$$

where b is another constant parameter. This model has been verified for 2-D Ising models. Similar formulation is intended to be used for the current application. However, in the present problem, $\lambda_{eff} \sim \frac{1}{N}$. Thus, it is impossible to provide a finite effective load per node to an infinite sized network, which means there can not be any finite load phase transition in an infinite-sized network, i.e., $\lambda^c_{eff}(\infty) = 0$ in the current configuration. Hence, for this case

$$\lambda^c_{eff}(N) = bN^{-\frac{1}{\nu}} \tag{11}$$

for a coefficient b and an exponent ν. Figure 6 shows a good fit of this proposed model with $b = 1.18$ and $\nu \approx 0.9$.

Next the critical exponent β is calculated for networks of different size as shown in Fig. 7. Similar exponent values

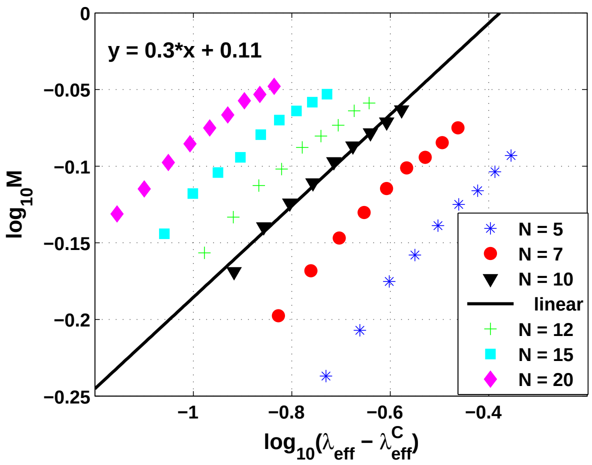

Fig. 7. Critical exponent β for networks of different Sizes

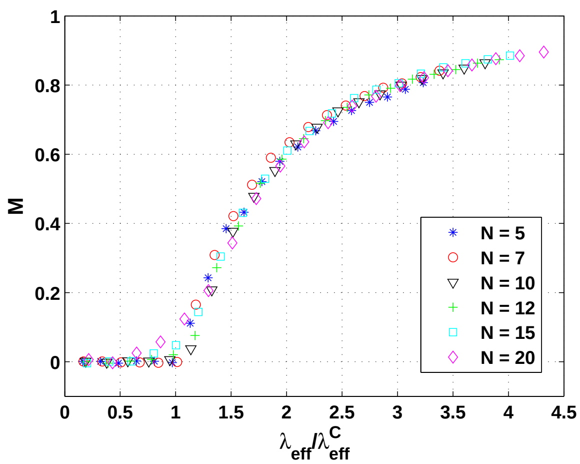

Fig. 8. Size independent global phase transition plot

are observed for different sizes, which is in agreement with the plots in Fig. 5. Figure 7 shows the possibility to obtain a size-independent global model for phase transition in square-grid communication networks. Such a model is achieved as shown in Fig. 8 by using a reduced effective load $\lambda_{red} \triangleq \frac{\lambda_{eff}}{\lambda_{eff}^{c}}$ instead of λ_{eff}. Usage of this correction factor draws inspiration from the discipline of Thermodynamics, where compressiblity curves for a variety of pure substances can be unified by in terms of a single parameter - reduced pressure or reduced temperature instead of a family of pressure or temperature points.

4. Effect of Routing Strategy

This section deals with the effect of the network routing strategy on the nature of phase transition. Thus far, all the results have been obtained with a deterministic routing strategy as explained earlier in Section 2. The results in this subsection have been generated with a change in the routing

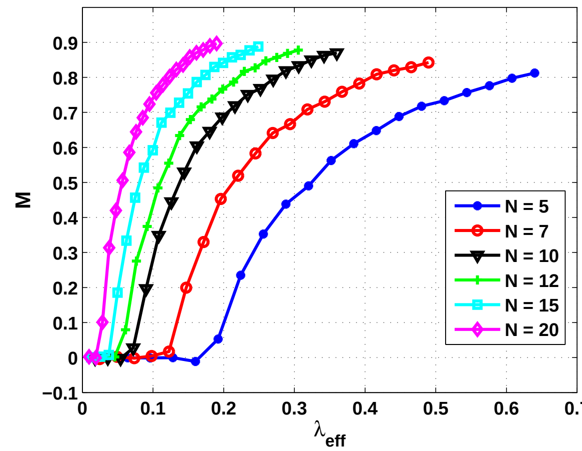

Fig. 9. Continuous phase transition plots for networks (with random routing) of different sizes

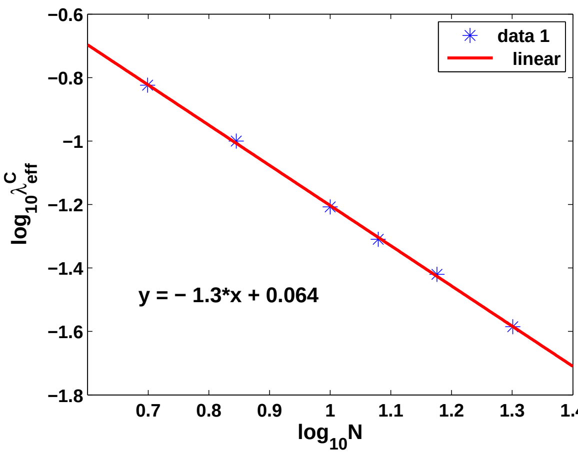

Fig. 10. Size dependency of critical network Load (for random routing)

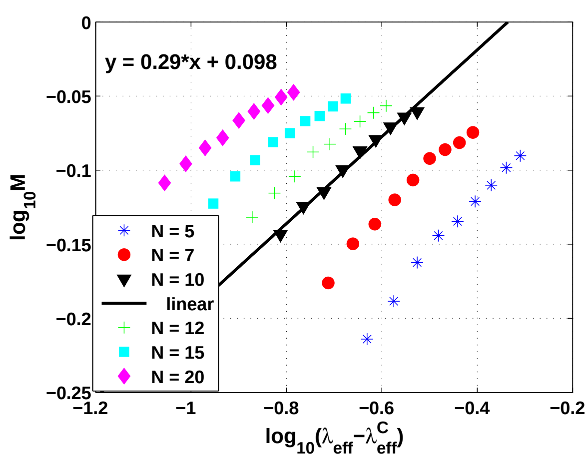

Fig. 11. Critical exponent β for networks (with random routing) of differ sizes

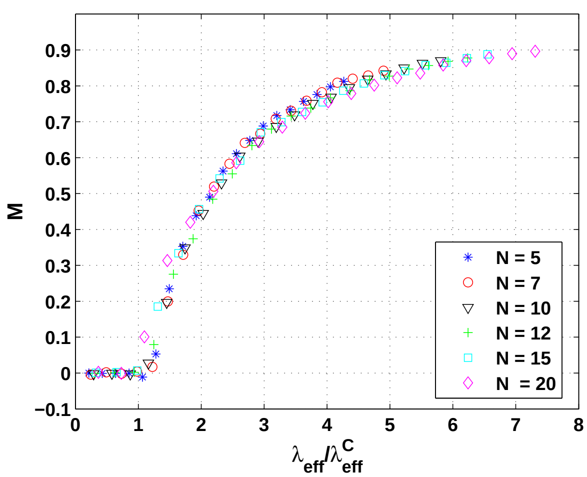

Fig. 12. Size independent global phase transition plot (for random routing)

strategy as explained below. In this routing strategy, a node is chosen randomly when two choices are available, whereas the previous strategy makes a deterministic choice of the node with the smaller queue length. Systematic study has been performed with this random routing strategy similar to that with the previous deterministic routing strategy. Figure 9 shows continuous phase transition plots for networks of different sizes with the random routing. As expected, the network becomes increasingly congested for decreased values of λ^c_{eff} that serves the normalization factor to generated the reduced effective load λ_{red} as explained in the previous section. Figure 10 shows size dependency of the critical network load with $b = 1.16$ and $\nu \approx 0.77$. Figure 11 shows the critical behavior for networks of different sizes with random routing. It is seen that the critical exponent $\beta \approx 0.29$ is largely insensitive to network size. As it was done for the previous routing strategy, a size-independent global phase transition model is identified for this random routing strategy as shown in Fig. 12.

5. Summary, Conclusions and Future work

A statistical mechanics-based method has been proposed for analysis of critical phenomena in communication networks. The basic concepts and underlying principles are validated by Monte Carlo simulation on the model of a wired communication network with single data type. The notions of order parameter, network-analog temperature are introduced for analysis of critical phenomena in the setting of statistical mechanics. A comprehensive analysis of finite-size scaling has been presented for a specific type of network structure and the effects of network routing strategies have been investigated. The results of Monte Carlo simulation show that the qualitative behavior of communication networks remains remarkably similar irrespective of the routing strategy although the coefficients and exponents in the power law models could be different for different routing strategies. This observation suggests correctness of the chosen global parameters for this statistical mechanics-based analysis and those could potentially be used as measures of performance and stability of communication networks. However, much theoretical and experimental research is needed before the statistical mechanics-based concept, presented in this paper, can be considered for analysis and synthesis of large-scale networks. A few examples are presented below as topics of future research.

- Investigation of the effects of packet arrival statistics, distribution of source and destination nodes.
- Investigation of the effects of network shapes (e.g., rectangular instead of square grid).
- Dynamic analysis of global stability and convergence in terms of the network equilibrium parameters (e.g. order parameter)
- Investigation of stability and performance robustness relative to exogenous disturbance (e.g. spatial variations in packet arrival distribution).
- Experimental validation on more complex and realistic (e.g., sensor network) topologies.

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